Finite difference methods for two-dimensional fractional dispersion equation

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Abstract

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance and other areas of application. In this paper we discuss a practical alternating directions implicit method to solve a class of two-dimensional initial-boundary value fractional partial differential equations with variable coefficients on a finite domain. First-order consistency, unconditional stability, and (therefore) first-order convergence of the method are proven using a novel shifted version of the classical Grünwald finite difference approximation for the fractional derivatives. A numerical example with known exact solution is also presented, and the behavior of the error is examined to verify the order of convergence.
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1. Introduction

Fractional derivatives are almost as old as their more familiar integer-order counterparts [30,40]. Fractional diffusion equations have recently been applied to many problems in physics (see an excellent review article by Metzler and Klafter [29]), finance [16,21,33,36,35], and hydrology [2,5,6,37,38]. Fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle plume spreads at a rate

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inconsistent with the classical Brownian motion model, and the plume may be asymmetric. When a fractional derivative replaces the second derivative in a diffusion or dispersion model, it leads to enhanced diffusion (also called superdiffusion). For fractional partial differential equations with constant coefficients, analytical solutions are available using Laplace–Fourier transform methods [3,11,23,25]. However, many practical problems require a model with variable coefficients [4,7,8,26]. This paper presents a practical numerical method for solving multi-dimensional fractional partial differential equations with variable coefficients, using a variation on the classical alternating-directions implicit (ADI) Euler method. We prove that this method, using a novel shifted version of the usual Grünwald finite difference approximation for the non-local fractional derivative operator, is first-order consistent and unconditionally stable for a fractional diffusion/dispersion equation with Dirichlet boundary conditions. A numerical example is included, along with its exact analytical solution, to validate the method and its order of convergence.

Consider the two-dimensional fractional diffusion (dispersion) equation

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} + e(x,y)\frac{\partial^\beta u(x,y,t)}{\partial y^\beta} + q(x,y,t) \tag{1}$$

on a finite rectangular domain $x_L < x < x_H$ and $y_L < y < y_H$, with $1 < \alpha \leqslant 2$ and $1 < \beta \leqslant 2$, $d(x,y) > 0$ and $e(x,y) > 0$, and assume that this fractional diffusion equation has a unique and sufficiently smooth solution under the following initial and boundary conditions (some results on existence and uniqueness are developed in [14]). Define the initial condition from $u(x,y,t=0) = f(x,y)$ for $x_L < x < x_H$, $y_L < y < y_H$, and Dirichlet boundary conditions $u(x,y,t) = B(x,y,t)$ on the boundary (perimeter) of the rectangular region $x_L \leqslant x \leqslant x_H, y_L \leqslant y \leqslant y_H$, with the additional restriction that $B(x_L,y,t) = B(x,y_L,t) = 0$. In physical applications, this means that the left/lower boundary is set far enough away from an evolving plume that no significant concentrations reach that boundary. The classical dispersion equation in two dimensions is given by $\alpha = \beta = 2$. The values of $1 < \alpha < 2$ or $1 < \beta < 2$ model a super-diffusive process in that direction.

Eq. 1 uses a Riemann fractional derivative of order α, defined by

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_L^x \frac{f(\xi)}{(x-\xi)^{\alpha+1-n}}d\xi, \tag{2}$$

where n is an integer such that $n - 1 < \alpha \leqslant n$. In most of the related literature, the case $L = 0$ is called the Riemann–Liouville form, and the case $L = -\infty$ is the Liouville definition for the fractional derivative. Fractional derivatives are nonlocal operators of convolution type [1,9,24]. The value of the fractional derivative at a point x depends on the function values at all the points in the interval $(-\infty, x)$. With our boundary conditions (and zero-extending the solution functions for $x < x_L$ or $y < y_L$) the Riemann and Liouville forms in (1) become equivalent. For more details on fractional derivative concepts and definitions, see [30,32,40].

An implicit Euler method for solving one-dimensional fractional differential equations is discussed in [26,28]. A shifted Grünwald finite difference scheme (3) is used to approximate the fractional space derivative in an implicit Euler method. Stability is proven for the implicit method, and also for an explicit Euler method under the condition that $\Delta t/\Delta x^\alpha$ is suitably bounded. It is proven that methods based on the unshifted Grünwald approximation are unstable. A different method for solving 1D fractional partial differential equations is pursued in the recent paper of Liu et al. [19]. They transform the partial differential equation into a system of ordinary differential equations (Method of Lines), which is then solved using backward differentiation formulas. Fix and Roop [15] and Ervin and Roop [13] develop finite element methods for certain 1D partial differential equations with constant coefficients on the fractional derivative terms. Ervin and Roop [14] extend this approach for multi-dimensional partial differential equations with constant coefficients on the fractional derivative terms. Deng et al. [10] discuss a numerical solution of a fractional advection–dispersion equation based on a three-point approximation for the fractional derivative. Lynch et al. [20] apply an explicit method and a related semi-implicit method to solve a one-dimensional anomalous dif-

fusion problem, with the boundary conditions specified on the left boundary. They estimate the fractional derivative using the L2 method proposed by Oldham and Spanier [31], in which the fractional derivative of order α is replaced by an $(\alpha - 2)$ fractional integral of the second derivative, the fractional integral is estimated by some variant of the Grünwald formula, and the second derivative is approximated by the standard three-point centered finite difference formula. Langlands and Henry [18] discuss similar numerical methods for time-fractional diffusion equations $\partial^\gamma u(x,t)/\partial t^\gamma = \partial^2 u(x,t)/\partial x^2$. Yuste and Acedo [41] prove stability of an explicit Euler method for the time-fractional diffusion equation under the condition that $\Delta t^\gamma/\Delta x^2$ is suitably bounded.

2. Numerical method

It was shown in [26] that using the usual (i.e., unshifted) Grünwald formula to discretize the one-dimensional dispersion equation results in an unstable finite difference scheme. Since the one dimensional case may be viewed as a special case of the two dimensional problem when $e(x,y,t) = 0$ (or $d(x,y,t) = 0$), we infer that the use of the usual Grünwald formula will produce an unstable method (a fact that can also be shown directly by an argument similar to Proposition 2.3 in [26]).

Therefore we start with a right-shifted Grünwald approximation to the fractional derivative term and in this paper we show that this leads to a stable (and convergent) alternating-directions implicit (ADI) implementation for the two-dimensional implicit Euler formulation. The right-shifted Grünwald formula for $1 < \alpha \leqslant 2$ is [26]

$$\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{N_x \to \infty} \frac{1}{h^\alpha} \sum_{k=0}^{N_x} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} u(x-(k-1)h, y, t), \tag{3}$$

where N_x is a positive integer, $h = (x - x_L)/N_x$ and $\Gamma(\cdot)$ is the gamma function. We also define the 'normalized' Grünwald weights by

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k \binom{\alpha}{k} \tag{4}$$

and remark that these normalized weights only depend on the order α and the index k. (For example, the first four terms of this sequence are given by $g_{\alpha,0} = 1$, $g_{\alpha,1} = -\alpha$, $g_{\alpha,2} = \alpha(\alpha-1)/2!$, $g_{\alpha,3} = -\alpha(\alpha-1)(\alpha-2)/3!$.)

For the numerical approximation scheme, define $t_n = n\Delta t$ to be the integration time $0 \leqslant t_n \leqslant T$, $\Delta x = h > 0$ is the grid size in x-direction, $\Delta x = (x_H - x_L)/N_x$, with $x_i = x_L + i\Delta x$ for $i = 0, \ldots, N_x$; $\Delta y > 0$ is the grid size in y-direction, $\Delta y = (y_H - y_L)/N_y$, with $y_j = y_L + j\Delta y$ for $j = 0, \ldots, N_y$. Define $u_{i,j}^n$ as the numerical approximation to $u(x_i, y_j, t_n)$. Similar notation gives $d_{i,j} = d(x_i, y_j)$, $e_{i,j} = e(x_i, y_j)$, $q_{i,j}^n = q(x_i, y_j, t_n)$. The initial conditions are set by $u_{i,j}^0 = f_{i,j} = f(x_i, y_j)$. The Dirichlet boundary condition at $x = x_L$ by $u_{0,j}^n = B_{0,j}^n = B(x_L, y_j, t_n)$, and similarly for the Dirichlet boundary conditions on the other three sides of the rectangular region.

If the shifted Grünwald estimates are substituted into the two-dimensional diffusion problem (1) to get the implicit Euler approximation, the resulting finite difference equations are

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{d_{i,j}}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} u_{i-k+1,j}^{n+1} + \frac{e_{i,j}}{(\Delta y)^\beta} \sum_{k=0}^{j+1} g_{\beta,k} u_{i,j-k+1}^{n+1} + q_{i,j}^{n+1}. \tag{5}$$

Eq. 5 may be written as

$$u_{i,j}^{n+1} - \frac{d_{i,j}\Delta t}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} u_{i-k+1,j}^{n+1} - \frac{e_{i,j}\Delta t}{(\Delta y)^\beta} \sum_{k=0}^{j+1} g_{\beta,k} u_{i,j-k+1}^{n+1} = u_{i,j}^n + q_{i,j}^{n+1}\Delta t. \tag{6}$$

Define the following fractional partial difference operators:

$$\delta_{\alpha,x}u_{i,j}^{n+1} = \frac{d_{i,j}}{(\Delta x)^{\alpha}} \sum_{k=0}^{i+1} g_{\alpha,k} u_{i-k+1,j}^{n+1},$$

which is an $O(\Delta x)$ approximation to the αth fractional derivative as shown in Theorem 2.4 in [26] (this assumes a zero boundary condition at the lower edge of the domain), and similarly,

$$\delta_{\beta,y}u_{i,j}^{n+1} = \frac{e_{i,j}}{(\Delta y)^{\beta}} \sum_{k=0}^{j+1} g_{\beta,k} u_{i,j-k+1}^{n+1}$$

is an $O(\Delta y)$ approximation to the βth fractional derivative term. With these operator definitions, the implicit Euler finite difference method may be written in the operator form

$$(1 - \Delta t\delta_{\alpha,x} - \Delta t\delta_{\beta,y})u_{i,j}^{n+1} = u_{i,j}^n + q_{i,j}^{n+1}\Delta t. \tag{7}$$

The above two-dimensional implicit Euler method has local truncation error of the form $O(\Delta t) + O(\Delta x) + O(\Delta y)$. In a manner similar to the proof for the one-dimensional case (Theorem 2.7 in [26], Theorem 3.3 in this paper), it can be shown to be unconditionally stable. Thus, according to the Lax Equivalence Theorem, the method is convergent. However, at each time step, this implicit formulation requires the solution of a very large non-sparse linear system of equations with $(N_x - 1)\cdot(N_y - 1)$ unknowns, which is computationally intensive. The problem becomes more computationally demanding as finer grid resolutions and/or higher spatial dimensions are considered.

One standard method in the classical multi-dimensional PDEs is the use of ADI methods, where the difference equations are specified and solved in one directions at a time. For the ADI methods (and in similar fashion for the splitting methods), the operator form is written in a directional separation product form

$$(1 - \Delta t\delta_{\alpha,x})(1 - \Delta t\delta_{\beta,y})u_{i,j}^{n+1} = u_{i,j}^n + q_{i,j}^{n+1}\Delta t, \tag{8}$$

which introduces an additional perturbation error equal to

$$(\Delta t)^2(\delta_{\alpha,x}\delta_{\beta,y})u_{i,j}^{n+1}. \tag{9}$$

Eq. 8 may be written in the matrix form

$$\underline{S}\,\underline{T}U^{n+1} = \underline{U}^n + \underline{R}^{n+1}, \tag{10}$$

where the matrices $\underline{S}$ and $\underline{T}$ represent the operators $(1 - \Delta t\delta_{\alpha,x})$ and $(1 - \Delta t\delta_{\beta,y})$, and

$$\underline{U}^n = [u_{1,1}^n, u_{2,1}^n, \ldots, u_{N_x-1,1}^n, u_{1,2}^n, u_{2,2}^n, \ldots, u_{N_x-1,2}^n, \ldots, u_{1,N_y-1}^n, u_{2,N_y-1}^n, \ldots, u_{N_x-1,N_y-1}^n]^{\mathrm{T}}$$

and the vector $\underline{R}^{n+1}$ absorbs the forcing term and the boundary conditions in the discretized equation.

Computationally, the ADI method for the above form is then set up and solved by the following iterative scheme. At time t_{n+1}:

(1) First solve the problem in the *x*-direction (for each fixed y_j) to obtain an intermediate solution $u_{i,j}^*$ from

$$(1 - \Delta t\delta_{\alpha,x})u_{i,j}^* = u_{i,j}^n + q_{i,j}^{n+1}\Delta t, \tag{11}$$

(2) then solve in the *y*-direction (for each fixed x_i)

$$(1 - \Delta t\delta_{\beta,y})u_{i,j}^{n+1} = u_{i,j}^*. \tag{12}$$

The initial and boundary conditions for the numerical solutions $u_{i,j}^{n+1}$ and $u_{i,j}^{n}$ are defined from the given initial and Dirichlet boundary conditions. Prior to carrying out step one of solving (11), the boundary conditions for the intermediate solution $u_{i,j}^{*}$ should be set from Eq. (12) (which incorporates the values of $u_{i,j}^{n+1}$ at the boundary). Otherwise the order of convergence will be adversely affected. Specifically, assume that the Dirichlet boundary conditions are given by the function $B(x, y, t)$ on the boundary of the rectangular region $x_L < x < x_H, y_L < y < y_H$. For example, on the right boundary we write $u_{N_x,j}^{n+1} = B_{N_x,j}^{n+1}$, and compute the boundary values for u^* from

$$u_{N_x,j}^* = (1 - \Delta t \delta_{\beta,y}) B_{N_x,j}^{n+1}$$

for use in setting up and solving the sets of equations defined by (11). See the proof of Theorem 3.3 for more details.

Below we show that the ADI for the implicit Euler method, as defined by (8) or (10), or equivalently by (11) and (12), is consistent and stable and therefore, by the Lax Equivalence theorem (p.45 in [34]), it is convergent.

To prove the consistency of the ADI-Euler method, note that the three difference operators used in (5) each have a local truncation error with $O(\Delta t)$, $O(\Delta x)$, and $O(\Delta y)$ respectively. The $O(\Delta t)$ for the time derivative term is obtained from the classical Taylor's expansion. The $O(\Delta x)$ and $O(\Delta y)$ for the local truncation error of the fractional derivative terms was proved in [26]. The only remaining term in the local error for the ADI-Euler method, is the additional perturbation error of (9). Theorem 3.1 below shows that $(\delta_{\alpha,x}\delta_{\beta,y})u_{i,j}^{n+1}$ converges to the mixed fractional derivative linearly, as $O(\Delta x) + O(\Delta y)$. Therefore the local truncation error of the ADI-Euler method (8) is $O(\Delta t) + O(\Delta x) + O(\Delta y)$.

3. Consistency and stability

In this section, we demonstrate that the alternating directions implicit Euler method is both consistent and unconditionally stable for the fractional initial-boundary value problem (1). We begin with the proof of consistency. As mentioned in the previous section, this depends on the mixed fractional derivative term. For any positive integer l, let $W^{l,1}(\mathbb{R}^2)$ denote the collection of all functions $f \in C^l(\mathbb{R}^2)$ whose partial derivatives up to order l are in $\mathscr{L}^1(\mathbb{R}^2)$ and whose partial derivatives up to order $l - 1$ vanish at infinity.

Define the Liouville form of a fractional derivative of some non-integer order $\alpha > 0$ by

$$\frac{\partial^\alpha}{\partial r^\alpha} f(r) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial r^n} \int_{-\infty}^{r} f(\xi)(r-\xi)^{n-\alpha-1} \mathrm{d}\xi, \tag{13}$$

where n is an integer such that $n - 1 < \alpha < n$. Then it easy to see that

$$\frac{\partial^\alpha}{\partial r^\alpha} f(r) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial r^n} \int_{0}^{\infty} \xi^{n-\alpha-1} f(r-\xi) \mathrm{d}\xi$$

and if $\mathscr{F}[f](k) = \hat{f}(k) = \int e^{ikx} f(x) \mathrm{d}x$ denotes the Fourier transform of some $\mathscr{L}^1$-function f then $\frac{\partial^\alpha}{\partial r^\alpha} f(r)$ has Fourier transform $(-ik)^\alpha \hat{f}(k)$ (see for example [40]). Since the mixed derivative $\frac{\partial^\beta \partial^\alpha}{\partial x^\alpha \partial y^\beta} f(x,y)$ has Fourier-transform $(-ik)^\alpha(-il)^\beta \hat{f}(k,l)$ it follows that

$$\frac{\partial^\beta}{\partial y^\beta} \frac{\partial^\alpha}{\partial x^\alpha} f(x,y) = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} f(x,y).$$

The one-dimensional fractional derivative can be approximated by the shifted Grünwald-formula

$$\frac{\partial^\alpha}{\partial r^\alpha} f(r) = h^{-\alpha} \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} f(r - (m-p)h) + O(h)$$

for any fixed integer $p \geqslant 0$. See [39] for the case $p = 0$ and [26] for the general case. The following result shows that an O(h) approximation also holds for the mixed fractional derivative.

For non-integers $\alpha, \beta > 0$, integers $p, q \geqslant 1$ and grid sizes $h_x, h_y > 0$ let

$$\Delta_{h_x,h_y;p,q}^{\alpha,\beta} f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \binom{\alpha}{n} \binom{\beta}{m} f(x - (n-p)h_x, y - (m-q)h_y) \tag{14}$$

define the shifted mixed Grünwald formula. The next result proves this formula is a first-order consistent approximation to the mixed fractional derivative. This implies that the additional perturbation error (9) for the ADI method is O($(h_x + h_y)\Delta t$) which is small compared to the approximation errors for the other terms in (5). Hence we do not sacrifice much accuracy when we replace the full implicit Euler scheme by a more efficient ADI method.

Theorem 3.1. *Let $r > \alpha + \beta + 3$ be an integer. Then for $f \in W^{r,1}(\mathbb{R}^2)$*

$$\frac{\partial^\beta}{\partial y^\beta} \frac{\partial^\alpha}{\partial x^\alpha} f(x,y) = h_x^{-\alpha} h_y^{-\beta} \Delta_{h_x,h_y;p,q}^{\alpha,\beta} f(x,y) + O(h_x + h_y) \tag{15}$$

uniformly in $(x,y) \in \mathbb{R}^2$.

Proof. We adapt the argument used in [39,26]. See also [27]. Let $\mathscr{F}[f](k,l) = \hat{f}(k,l) = \int_{\mathbb{R}^2} e^{i(kx+ly)} f(x,y) \, dx \, dy$ denote the Fourier transform of $f \in \mathscr{L}^1(\mathbb{R}^2)$. Note that for any $a, b \in \mathbb{R}$ we have

$$\mathscr{F}\big[f(x-a, y-b)\big](k,l) = e^{iak} e^{ibl} \hat{f}(k,l).$$

Moreover, we have the well known result that for any $\gamma > 0$

$$(1+z)^\gamma = \sum_{m=0}^{\infty} \binom{\gamma}{m} z^m \tag{16}$$

is absolute convergent for $|z| \leqslant 1$.

It is readily verified that

$$\int_{\mathbb{R}^2} \left| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \binom{\alpha}{n} \binom{\beta}{m} f(x-(n-p)h_x, y-(m-q)h_y) \right| dx \, dy$$

$$\leqslant \|f\|_1 \left(\sum_{n=0}^{\infty} \left| \binom{\alpha}{n} \right| \right) \left(\sum_{m=0}^{\infty} \left| \binom{\beta}{m} \right| \right) < \infty.$$

Consequently, the right-hand side of (14) defines an element of $\mathscr{L}^1(\mathbb{R}^2)$.

Thus we can take the Fourier transform in (14) to obtain

$$\mathscr{F}\left[h_x^{-\alpha} h_y^{-\beta} \Delta_{h_x,h_y;p,q}^{\alpha,\beta} f \right](k,l) = h_x^{-\alpha} h_y^{-\beta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \binom{\alpha}{n} \binom{\beta}{m} e^{ik(n-p)h_x} e^{il(m-q)h_y} \hat{f}(k,l)$$

$$= \left(h_x^{-\alpha} e^{-ikph_x} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} (e^{ikh_x})^n \right) \left(h_y^{-\beta} e^{-ilqh_y} \sum_{m=0}^{\infty} (-1)^m \binom{\beta}{m} (e^{ilh_y})^m \right) \hat{f}(k,l)$$

$$= e^{-ikph_x} h_x^{-\alpha} \left(1 - e^{ikh_x} \right)^\alpha e^{-ilqh_y} h_y^{-\beta} \left(1 - e^{ilh_y} \right)^\beta \hat{f}(k,l)$$

$$= (-ik)^\alpha (-il)^\beta \omega_{\alpha,p}(-ikh_x) \omega_{\beta,q}(-ilh_y) \hat{f}(k,l),$$

where

$$\omega_{\alpha,p}(z) = \mathrm{e}^{pz}\left(\frac{1-\mathrm{e}^{-z}}{z}\right)^{\alpha} \quad \text{and} \quad \omega_{\beta,q}(z) = \mathrm{e}^{qz}\left(\frac{1-\mathrm{e}^{-z}}{z}\right)^{\beta}.$$

Hence

$$\mathscr{F}\left[h_x^{-\alpha}h_y^{-\beta}\Delta_{h_x,h_y;p,q}^{\alpha,\beta}f\right](k,l) - \mathscr{F}\left[\frac{\partial^{\beta}}{\partial y^{\beta}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}f\right](k,l) = (-\mathrm{i}k)^{\alpha}(-\mathrm{i}l)^{\beta}\big(\omega_{\alpha,p}(-\mathrm{i}kh_x)\omega_{\beta,q}(-\mathrm{i}lh_y)-1\big)\hat{f}(k,l)$$

$$= \hat{\varphi}(h_x,h_y;k,l).$$

Observe that for $\gamma > 0$ there exists a constant $C = C(\gamma) > 0$ such that

$$\left|\left(\frac{1-\mathrm{e}^{\mathrm{i}x}}{-\mathrm{i}x}\right)^{\gamma} - 1\right| \leqslant C|x| \quad \text{for all } x \in \mathbb{R}. \tag{17}$$

Moreover, since

$$\big|\omega_{\alpha,p}(-\mathrm{i}x)\omega_{\beta,q}(-\mathrm{i}y)-1\big| \leqslant \big|\omega_{\beta,q}(-\mathrm{i}y)\big|\big|\omega_{\alpha,p}(-\mathrm{i}x)-1\big| + \big|\omega_{\beta,q}(-\mathrm{i}y)-1\big|$$

and $|\omega_{\beta,q}(-\mathrm{i}y)| \leqslant C$ for some constant $C > 0$ and all $y \in \mathbb{R}$, we get from (17) that

$$\big|\omega_{\alpha,p}(-\mathrm{i}x)\omega_{\beta,q}(-\mathrm{i}y)-1\big| \leqslant C_1(|x|+|y|) \leqslant C_2\|(x,y)\|_2,$$

where $\|(x,y)\|_2$ denotes the Euclidean norm.

In view of the inequality $\|(kh_x,lh_y)\|_2 \leqslant (h_x+h_y)\|(k,l)\|_2$ we therefore get

$$|\hat{\varphi}(h_x,h_y;k,l)| \leqslant C(h_x+h_y)|k|^{\alpha}|l|^{\beta}\|(k,l)\|_2|\hat{f}(k,l)| \leqslant C(h_x+h_y)\big(1+\|(k,l)\|_2\big)^{\alpha+\beta+1}|\hat{f}(k,l)|. \tag{18}$$

Moreover, since $f \in W^{r,1}(\mathbb{R}^2)$ the Riemann–Lebegue lemma implies that for some constant $M > 0$

$$|\hat{f}(k,l)| \leqslant M(1+\|(k,l)\|_2)^{-r}$$

and hence

$$|\hat{\varphi}(h_x,h_y;k,l)| \leqslant C(h_x+h_y)(1+\|(k,l)\|_2)^{\alpha+\beta+1-r}.$$

Since the right hand side of this inequality is integrable we conclude from Fourier inversion that

$$\left|h_x^{-\alpha}h_y^{-\beta}\Delta_{h_x,h_y;p,q}^{\alpha,\beta}f(x,y) - \frac{\partial^{\beta}}{\partial y^{\beta}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}f(x,y)\right| = \left|c_2\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(xk+yl)}\hat{\varphi}(h_x,h_y;k,l)\mathrm{d}k\,\mathrm{d}l\right|$$

$$\leqslant c_2\int_{\mathbb{R}^2}\big|\hat{\varphi}(h_x,h_y;k,l)\big|\mathrm{d}k\,\mathrm{d}l$$

$$\leqslant C_1(h_x+h_y)\int_{\mathbb{R}^2}(1+\|(k,l)\|_2)^{\alpha+\beta+1-r}\mathrm{d}k\,\mathrm{d}l = C_2(h_x+h_y)$$

independent of $(x,y) \in \mathbb{R}^2$ and the proof is complete. $\square$

The stability proof is based on showing that each one-dimensional system is unconditionally stable. The argument is similar to Theorem 2.7 in [26].

Theorem 3.2. *Each one-dimensional implicit system defined by the linear difference Eqs.* (11) *and* (12) *is unconditionally stable for all* $1 < \alpha < 2$.

Proof. At each gridpoint y_k, for $k = 1, \ldots, N_y - 1$, consider the linear system of equations defined by (11). This system of equations may be written as $\underline{A}_k\underline{U}_k^* = \underline{U}_k^n + \Delta t\underline{Q}_k^{n+1}$ where, incorporating the boundary conditions from (12), we have

$$\underline{U_k}^* = [u_{1,k}^*, u_{2,k}^*, \ldots, u_{N_x-1,k}^*]^{\mathrm{T}},$$
$$\underline{U_k}^n + \Delta t \underline{Q_k}^{n+1} = [u_{1,k}^n + q_{1,k}^{n+1}\Delta t, u_{2,k}^n + q_{2,k}^{n+1}\Delta t, \ldots, u_{N_x-1,k}^n + q_{N_x-1,k}^{n+1}\Delta t + D_{N_x-1,k}g_{\alpha,0}(1 - \Delta t\delta_{\beta,y})B_{N_x,k}^{n+1}]^{\mathrm{T}}$$

and $\underline{A_k} = [A_{i,j}]$ is the $(N_x - 1) \times (N_x - 1)$ matrix of coefficients resulting from the system of difference equations at the gridpoint y_k, where the matrix entries along the ith row are defined from (11). For example, for $i = 1$ the equation becomes

$$-D_{1,k}g_{\alpha,2}u_{0,k}^* + (1 - D_{1,k}g_{\alpha,1})u_{1,k}^* - D_{1,k}g_{\alpha,0}u_{2,k}^* = u_{1,k}^n + q_{1,k}^{n+1}\Delta t$$

for $i = 2$ we have

$$-D_{2,k}g_{\alpha,3}u_{0,k}^* - D_{2,k}g_{\alpha,2}u_{1,k}^* + (1 - D_{2,k}g_{\alpha,1})u_{2,k}^* - D_{2,k}g_{\alpha,0}u_{3,k}^* = u_{2,k}^n + q_{2,k}^{n+1}\Delta t$$

and for $i = N_x - 1$ we get

$$-D_{N_x-1,k}g_{\alpha,N_x}u_{0,k}^* - D_{N_x-1,k}g_{\alpha,N_x-1}u_{1,k}^* + \cdots + (1 - D_{N_x-1,k}g_{\alpha,1})u_{N_x-1,k}^* - D_{N_x-1,k}g_{\alpha,0}u_{N_x,k}^* = u_{N_x-1,k}^n + q_{N_x-1,k}^{n+1},$$

where the coefficients

$$D_{i,k} = \frac{d_{i,k}\Delta t}{(\Delta x)^\alpha}.$$

Therefore the resulting matrix entries $A_{i,j}$ for $i = 1, \ldots, N_x - 1$ and $j = 1, \ldots, N_x - 1$ are defined by

$$A_{i,j} = \begin{cases} -D_{i,k}g_{\alpha,i-j+1} & \text{for } j \leqslant i - 1, \\ 1 - D_{i,k}g_{\alpha,1} & \text{for } j = i, \\ -D_{i,k}g_{\alpha,0} & \text{for } j = i + 1, \\ 0 & \text{for } j > i + 1. \end{cases}$$

We will now apply the Greshgorin theorem (cf. [17], pp. 135–136) to conclude that every eigenvalue of the matrix $\underline{A_k}$ has a magnitude strictly larger than 1.

Note that $g_{\alpha,1} = -\alpha$, and for $1 < \alpha < 2$ and $i \neq 1$ we have $g_{\alpha,i} > 0$. Substituting $z = -1$ into (16) yields $\sum_{i=0}^{\infty}g_{\alpha,i} = 0$, and then it follows that $-g_{\alpha,1} > \sum_{k=0,k\neq1}^{k=N}g_{\alpha,i}$ for any $N > 1$. According to the Greschgorin theorem, the eigenvalues of the matrix $\underline{A_k}$ are in the disks centered at $A_{i,i} = 1 - D_{i,k}g_{\alpha,1} = 1 + D_{i,k}\alpha$, with radius

$$r_i = \sum_{l=1,l\neq i}^{N_x-1} |A_{i,l}| \leqslant \sum_{l=1,l\neq i}^{i+1} D_{i,k}g_{\alpha,i-l+1} < D_{i,k}\alpha.$$

Hence every eigenvalue λ of the matrix $\underline{A_k}$ has a real part larger than 1, and therefore a magnitude larger than 1. Hence, the spectral radius of each matrix $\underline{A_k}^{-1}$ is less than one. This proves that the method is stable (cf. [42], pp. 13–15).

Similar results hold for the finite difference equations defined by (12). When sweeping in the alternate direction (i.e., with the x_k grid point fixed) to solve for u^{n+1} from u^*, the resulting system is then defined by $\underline{C_k}\underline{U_k}^{n+1} = \underline{U_k}^*$, where

$$\underline{U_k}^* = [u_{k,1}^*, u_{k,2}^*, \ldots, u_{k,N_y-1}^*]^{\mathrm{T}},$$
$$\underline{U_k}^{n+1} = [u_{k,1}^{n+1}, u_{k,2}^{n+1}, \ldots, u_{k,N_y-1}^{n+1}]^{\mathrm{T}}$$

and $\underline{C_k} = [C_{i,j}]$ is the matrix of coefficients resulting from the system of difference equations at the gridpoint x_k for $k = 1, \ldots, N_x - 1$. (Note that the $\underline{U_k}^*$ and $\underline{U_k}^{n+1}$ are defined differently than the solution vectors in

the previous sweep direction due to re-arrangement of gridpoints in the y-direction.) The entries of the matrix $\underline{C}_k$ are defined from (12), for $i = 1, \ldots, N_y - 1$ and $j = 1, \ldots, N_y - 1$ by

$$C_{i,j} = \begin{cases} -E_{k,i} g_{\beta,i-j+1} & \text{for } j \leqslant i - 1, \\ 1 - E_{k,i} g_{\beta,1} & \text{for } j = i, \\ -E_{k,i} g_{\beta,0} & \text{for } j = i + 1, \\ 0 & \text{for } j > i + 1, \end{cases}$$

where the coefficients

$$E_{k,i} = \frac{e_{k,i} \Delta t}{(\Delta y)^\beta}.$$

Similar arguments show that each eigenvalue of of the matrix $\underline{C}_k$ has a real part (hence, also a magnitude) strictly larger than one. Therefore, the spectral radius $\rho(\underline{C}_k^{-1}) < 1$, and hence this system is also unconditionally stable. $\quad\square$

Next we show that, if the operators $(1 - \Delta t \delta_{\alpha,x})$ and $(1 - \Delta t \delta_{\beta,y})$ in (8) commute, the ADI-Euler method is unconditionally stable. The requirement for the commutativity of these two operators is also a common assumption in establishing stability/convergence of the ADI methods in the classical (i.e., $\alpha = \beta = 2$) two-dimensional diffusion equation (see for example [12]). Note that the commutativity of these operators means that the matrices $\underline{S}$ and $\underline{T}$ in (10) commute. For example, if the diffusion coefficients are of the form $d = d(x)$ and $e = e(y)$, then these operators (matrices) commute.

We also remark that the ADI-Euler formulation of (10) is useful for the theoretical analysis of the method, while the formulation according to (11) and (12) is used in the actual computer implimentation.

The matrix $\underline{S}$ is a block diagonal matrix of $(N_y - 1) \times (N_y - 1)$ blocks whose blocks are the square $(N_x - 1) \times (N_x - 1)$ super-triangular $\underline{A}_k$ matrices resulting from Eq. (11). We may write $\underline{S} = \text{diag}(\underline{A}_1, \underline{A}_2, \ldots, A_{N_y-1})$.

The matrix $\underline{T}$ is a block super-triangular matrix of $(N_y - 1) \times (N_y - 1)$ blocks whose non-zero blocks are the square $(N_x - 1) \times (N_x - 1)$ diagonal matrices resulting from Eq. (12). That is, we may write $\underline{T} = [\underline{T}_{i,j}]$, where each $\underline{T}_{i,j}$ is an $(N_x - 1) \times (N_x - 1)$ matrix, such that for $j > i + 1$ $\underline{T}_{i,j} = 0$, and for $j \leqslant i + 1$ each $\underline{T}_{i,j}$ is a diagonal matrix $\underline{T}_{i,j} = \text{diag}((\underline{C}_1)_{i,j}, (\underline{C}_2)_{i,j}, \ldots, (C_{N_x-1})_{i,j})$, where the notation $(\underline{C}_k)_{i,j}$ refers to the (i,j)th entry of the matrix $\underline{C}_k$ defined previously.

Theorem 3.3. *The ADI-Euler method, defined by (10), is unconditionally stable for $1 < \alpha < 2$, $1 < \beta < 2$ if the matrices $\underline{S}$ and $\underline{T}$ commute.*

Proof. Since $\underline{S} = \text{diag}(\underline{A}_1, \ldots, \underline{A}_2, A_{N_y-1})$, the eigenvalues of the matrix $\underline{S}$ are in the union of the Greschgorin disks for the matrices $\underline{A}_k$'s. Applying the argument of Theorem 3.2, it follows that every eigenvalue of the matrix $\underline{S}$ has a real-part (and a magnitude) larger than 1. Therefore, the magnitude of every eigenvalue of the inverse matrix $\underline{S}^{-1}$ is less than 1, and hence the spectral radius of the matrix $\underline{S}^{-1}$ is less than 1.

Similarly, the eigenvalues of the matrix $\underline{T}$ are in the union of the Greschgorin disks for the matrices $\underline{C}_k$'s. Again the argument of Theorem 3.2 may be applied to show that the spectral radius of the matrix $\underline{T}^{-1}$ is less than 1.

Note that Eq. (10) implies that an error $\underline{\epsilon}^0$ in $\underline{U}^0$ results in an error $\underline{\epsilon}^n$ at time t_n in $\underline{U}^n$ given by

$$\underline{\epsilon}^n = (\underline{S}\,\underline{T})^{-n} \underline{\epsilon}^0.$$

Since matrices $\underline{S}$ and $\underline{T}$ commute, we may write the above equation as

$$\underline{\epsilon}^n = \underline{S}^{-n} \underline{T}^{-n} \underline{\epsilon}^0.$$

Since the spectral radius of each matrix $\underline{S}^{-1}$ and $\underline{T}^{-1}$ is less than one, it follows that $\underline{S}^{-n} \to \underline{0}$ and $\underline{T}^{-n} \to \underline{0}$ as $n \to \infty$, where $\underline{0}$ is the zero (or null) matrix (see Theorem 1.4 in [42]). Therefore the ADI-Euler method is stable. $\square$

Since the ADI-Euler method is consistent and unconditionally stable, the numerical solution produced by the ADI-Euler method converges to the exact solution, and this convergence is $O(\Delta x + \Delta y + \Delta t)$. But note that, in order to obtain this rate of convergence, the boundary conditions for the intermediate solution $\underline{U}^*$ should be set according to (12), prior to solving the system in step one of the numerical algorithm defined by (11), as detailed in the proof of Theorem 3.2.

Remark 3.4. The system matrices (11) and (12) are super-triangular. That is, these matrices are the sum of a super-diagonal matrix and a lower triangular matrix. In the numerical implementation, these systems can be efficiently solved by first performing a backward sweep to reduce the system to a lower triangular system. This is then followed by a forward (explicit) sweep to solve a triangular system to obtain the solution. The computational work is approximately equivalent to solving two triangular systems. To solve a triangular $n \times n$ matrix, $n(n+1)/2$ arithmetic operations (multiplication or division flops) are required [17]. So the computational work (flops) is approximately $n(n+1)$ operations. Note that, this is significantly less computations than a full Gaussian elimination operation which requires $n^3/2 + O(n^2)$ arithmetic operations. We also note that an operational count shows that approximately $N_x N_y (N_x + N_y + 1)$ flops are needed to solve the collections of these super-triangular systems to advance the numerical solution from time level t_n to t_{n+1} for this ADI method. This compares very favorably with approximately $(N_x N_y)^3/2$ arithmetic operations needed by the Gaussian elimination to solve the full $(N_x N_y) \times (N_x N_y)$ matrix that results from the implicit Euler discretization.

Remark 3.5. Implicit Euler methods may be preferable to explicit methods due to their unconditional stability. Although the explicit methods are faster for a given step size in time, it is usually necessary to use a much smaller step size Δt to maintain stability. The condition for stability of an explicit Euler solution to the space-fractional diffusion equation $\partial u/\partial t = \partial^\alpha u/\partial x^\alpha$ is that $\Delta t/\Delta x^\alpha < 1/\alpha$, see [28]. Stability of the explicit Euler method for the time-fractional diffusion equation $\partial^\gamma u/\partial t^\gamma = \partial^2 u/\partial x^2$ requires that $\Delta t^\gamma/\Delta x^2 < C_\gamma$, see [18]. It would be interesting to compare the implicit methods of this paper against a multivariable version of the explicit methods in [20]. It would also be useful to extend the stability results for explicit Euler methods to space–time fractional diffusion equations $\partial^\gamma u/\partial t^\gamma = \partial^\alpha u/\partial x^\alpha$, and one suspects that stability for such methods will require $\Delta t^\gamma/\Delta x^\alpha$ to be suitably bounded. Another efficient method is to approximate the fractional derivative by the first few terms of the Grünwald approximation [10]. This method should be used with caution. First of all, the remaining terms of the Grünwald approximation are only negligable when α is near an integer value. Second and perhaps more important in applications, truncating the Grünwald approximation results in a method that is not mass-preserving.

4. A numerical example

The fractional differential equation

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{1.8}u(x,y,t)}{\partial x^{1.8}} + e(x,y)\frac{\partial^{1.6}u(x,y,t)}{\partial y^{1.6}} + q(x,y,t)$$

was considered on a finite rectangular domain $0 < x < 1$, $0 < y < 1$, for $0 \leqslant t \leqslant T_{\text{end}}$ with the diffusion coefficients

$$d(x,y) = \Gamma(2.2)x^{2.8}y/6 = 0.18363375x^{2.8}y$$

Table 1
Maximum error behavior versus gridsize reduction for the example problem at time $T_{end} = 1$

Δt	$\Delta x = \Delta y$	Maximum error	Error rate
0.10000	0.10000	0.00126285	–
0.05000	0.05000	0.000673253	1.88
0.02500	0.02500	0.000348242	1.93
0.01250	0.01250	0.000176599	1.97
0.00625	0.00625	0.0000881217	2.00

and

$$e(x,y) = 2xy^{2.6}/\Gamma(4.6) = 0.1494624672xy^{2.6}$$

and the forcing function

$$q(x,y,t) = -(1 + 2xy)e^{-t}x^3y^{3.6}$$

with the initial conditions

$$u(x,y,0) = x^3y^{3.6}$$

and Dirichlet boundary conditions on the rectangle in the form $u(0,y,t) = u(x,0,t) = 0$, $u(1,y,t) = e^{-t}y^{3.6}$, and $u(x,1,t) = e^{-t}x^3$ for all $t \geqslant 0$.

The exact solution to this two-dimensional fractional diffusion equation is given by

$$u(x,y,t) = e^{-t}x^3y^{3.6},$$

which may be verified by direct differentiation and substitution in the fractional differential equation, using the formula

$$\frac{\partial^\alpha}{\partial x^\alpha}[x^p] = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}x^{p-\alpha}$$

for this Riemann–Liouville fractional derivative (2) with $L = 0$.

Table 1 shows the maximum absolute numerical error, at time $t = 1.0$, between the exact analytical solution and the numerical solution obtained by applying the ADI-Euler method discussed in this paper. The algorithm was implemented using the Intel Fortran compiler on a Dell Pentium PC. All computations were performed in single precision. The last column of the figure shows the order of the convergence of the method as the grid is refined (as all step sizes are halved), which is computed as the ratio of the maximum absolute error at the previous larger grid size to the current grid size. The (almost) linear reduction in the maximum error is observed, as expected from the order $O(\Delta t) + O(\Delta x) + O(\Delta y)$ of the convergence of the method.

Note that this example problem does not meet the requirement for the commutativity of the operators in (8) which was used to establish the stability of the ADI-Euler method. The linear convergence of the numerical solution for this example suggests that the stability results may be extended beyond the requirement for commutativity.

5. Conclusions

Two-dimensional fractional order partial differential equations may be solved by an implicit alternating directions method. If a shifted version of the Grünwald finite difference approximation formula for fractional derivatives is used in an implicit Euler method, then the resulting ADI method is unconditionally

stable and converges linearly. The method is unstable if the usual Grünwald formula is used. Additionally, to obtain the linear convergence, the boundary conditions for the intermediate solution must be treated carefully.

Acknowledgments

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